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HEAT CONDUCTION AND HEAT EXCHANGE IN TECHNOLOGICAL PROCESSES

ANALYTICAL METHOD OF CALCULATING TEMPERATURE DIFFERENCES FOR RECTANGULAR HEAT-RELEASING ELEMENTS IN THE BODY OF A HOMOGENEOUS HEAT-CONDUCTING RECTANGULAR PLATE WITH HEAT REMOVAL FROM ITS EDGES

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An analytical method has been presented for calculating average temperature differences for rectangular heat-releasing elements located in the body of a rectangular homogeneous thin heat-conducting plate relative to its edges with the same temperature. A stationary problem for the cases of heat removal from one, two, three, and four edges of such a plate has been considered.

Introduction. A number of structures of printed circuit boards have recently come into being, in which heat removal from electroradioelements (EREs) occurs mainly or solely by conduction through metal substrates or cores. Such structures of printed circuit boards are used in both traditional electronic devices and novel, recently designed frame constructions [1, 2]. In the first case heat removal from metal cores to the device casing is realized from one, two, or three core edges using so-called thermal connectors, and in the second case it is effected from four edges of the metal substrate of the printed circuit board by a tight coupling with the heat-removal panels or metal frames.

Formulation of the Problem. In a certain group of structures employing printed circuit boards with metal plates of substrates or cores, the cooling system of the entire device provides practically the same temperature at the edges of such plates. In the current study, consideration is given only to such conditions of heat removal for these plates.

The design of printed circuit boards with predominant conductive heat removal practically always involves the problem of determining the temperature difference for a steady regime between the casing of each ERE and the substrate and core edges from which heat is removed. Here, the typical particular problem is determination of the temperature difference for a steady regime between an element in the body of the plate of the metal substrate or core with boundaries of projection of each ERE on this plate and its heat-removing edges. Five basic (1–5) and three auxiliary (6–8) versions of models for this plate are presented in Fig. 1. In actual design, this particular problem is repeatedly solved in the optimization of the heat-removal system of the entire device as well as in the "thermal arrangement" of specific EREs on specific printed circuit boards [3]. Therefore, along with possessing acceptable accuracy, program implementations of its solution should be high-speed. In order to solve such problems, extensive use is made of numerical methods, specifically, finite-element methods. However, while providing high computation accuracy for complex objects, their program implementations are generally low-speed. Therefore, here it is often preferable to use analytical methods [4] that are long known and whose current development is presented in [5, 6].

The described method of calculating the temperature difference suggests homogeneity (the same thickness and the absence of holes) of the substrate or core of the printed circuit board. As for actual structures, the metal substrate or core (bounded rectangular plate) has certain inhomogeneities (holes and/or thickenings) in peripheral regions of the substrate outside the field of the ERE arrangement. However, there is an approach which makes it possible to consider

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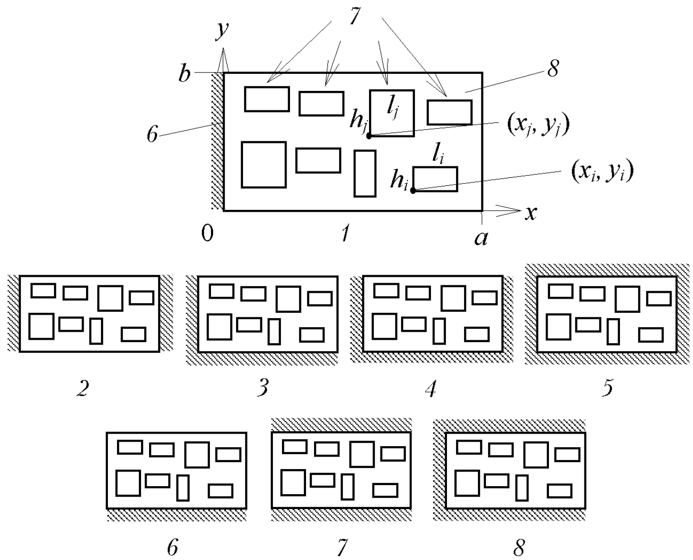

Fig. 1. Versions of heat removal from the plate: 1–5, basic versions; 6, heat-removal edge; 7, heating or passive elements; 8, metal plate.

(under some assumptions) an inhomogeneous rectangular plate as a homogeneous rectangular one with altered dimensions. The metal substrate or core of the printed circuit board are generally rather thin. Therefore, the temperature differences are calculated through solving a two-dimensional problem, and the obtained values of the temperature differences at each point of the plate are taken to be an average over the actual plate thickness. Here, the density of the thermal power generated by a single ERE at the point (x, y, z) of the volume of the actual plate is replaced by the surface density of the thermal power generated at the point (x, y) and equal to the ratio of the volumetric density to the actual plate thickness Z.

Usually EREs are placed on the printed circuit board such that their projections on it do not overlap. Moreover, there is necessarily a gap between their projections. Below, consideration is given only to this case. If N EREs $(N > 1)$ are placed on the printed circuit board, then, using the principle of superposition of thermal fields, the average temperature difference ΔT_i for each ith heating element in the body of the plate relative to the heat-removing edges can be represented by the sum of the average temperature differences ΔT_{ij} over the region of the ith element from all elements (including the ith one), which are calculated under the condition that only one ith or jth element is present on the printed circuit board:

$$\Delta T_i = \sum_{j=1}^{N} \Delta T_{ij} \, . \tag{1}$$

To determine ΔT_i, initially the method is described for calculating the temperature difference $u(x, y)$ for any point of the plate with the presence, in its body, of a single rectangular heat source with boundaries of the projection of an ERE on the plate, and subsequently, consideration is given to the summation of thermal fields from all N such sources for each of them with averaging over its area. These problems are solved in general form for one-, two-, three-, and four-way conductive heat removal, and in closing, specific solutions of each of these variants are presented.

Determination of Temperature Differences at the Points of a Homogeneous Rectangular Plate from a Single Rectangular Heat Source. The temperature distribution from a single heat source located in the body of a homogeneous heat-conducting rectangular plate can be described by the Poisson equation [7] taking for this case the form

$$\frac{\partial^2 u\,(x, y)}{\partial x^2} + \frac{\partial^2 u\,(x, y)}{\partial y^2} = -\,\rho\,(x, y)\,, \quad 0 \le x \le a\,, \quad 0 \le y \le b\,. \tag{2}$$

Here, $u(x, y)$ is the temperature difference at the point (x, y) from the ith rectangular heat source with dimensions $l_i \times h_i$ and $\rho(x, y)$ is the surface density of thermal power generated by the ith ERE at the point (x, y) of the plate with dimensions $a \times b$ divided into the specific thermal conductivity λ of the plate material [4]. Further on it is assumed that $\rho(x, y)$ = const $\neq 0$ within the boundaries $l_i \times h_i$ of the rectangular heat source (projections of the ERE casing on the plate), and beyond these boundaries, $\rho(x, y) = 0$. Then, the function $\rho(x, y)$ is of the form $\rho(x, y) = P_i/(l_i h_i Z\lambda)$ for $x_i \leq x \leq x_i + l_i$ and $y_i \leq y \leq y_i + h_i$ and $\rho(x, y) = 0$ for all other values of x and y, where (x_i, h_i) is the position of the left lower angle of the rectangular heat source, and P_i and $l_i h_i$ are respectively the power and the area of the rectangular ith heat source.

To solve Eq. (2), it is convenient to use the integral transform of Fourier functions in the Cartesian coordinate system with finite limits 0 and b [5] with respect to the variable y. Here, the general form of the nth transform $\overline{u_n}(x)$ of the function $u(x, y)$ for various combinations of the boundary conditions in Fig. 1 can be written (using Table 11 from [5]) as

$$\overline{u_n}(x) = \int_0^b u(x, y)\, \varphi_n(y)\, dy,$$

where $\varphi_n(y)$ is the kernel of the integral transformation, determined by a specific combination of the 1st- and 2nd-kind conditions for the versions of heat removal from the plate presented in Fig. 1. Here (in accordance with Table 11 from [5]), the following three types of combinations of the boundary conditions are possible for two boundaries $y = 0$ and $y = b$ with $0 \leq x \leq a$.

1. On the boundaries $y = 0$ and $y = b$, the 1st-kind boundary conditions ($u(x, y)|_{y=0} = u(x, y)|_{y=b} = 0$) are versions 5, 7, and 8 in Fig. 1. Here,

$$\varphi_n^{(1)}(y) = \sin\left(\mu_n^{(1)} y\right), \quad \mu_n^{(1)} = \frac{n\pi}{b}, \quad n = 1, 2, ..., \infty; \tag{3}$$

$$u(x, y) = \frac{2}{b} \sum_{n=1}^{\infty} \overline{u_n}(x) \sin\left(\mu_n^{(1)} y\right). \tag{4}$$

2. On the boundaries $y = 0$ and $y = b$, the 2nd-kind boundary conditions $\left(\left.\frac{du(x, y)}{dy}\right|_{y=0} = \left.\frac{du(x, y)}{dy}\right|_{y=b} = 0\right)$ are versions 1 and 2 in Fig. 1. Here,

$$\varphi_n^{(2)}(y) = \cos\left(\mu_n^{(2)} y\right), \quad \mu_n^{(2)} = \frac{n\pi}{b}, \quad n = 1, 2, ..., \infty; \tag{5}$$

$$u(x, y) = \frac{\overline{u_0}(x)}{b} + \frac{2}{b} \sum_{n=1}^{\infty} \overline{u_n}(x) \cos\left(\mu_n^{(2)} y\right). \tag{6}$$

3. On the boundary $y = 0$, the 1st-kind boundary condition, and on the boundary $y = b$, the 2nd-kind boundary condition $\left(u(x, y)|_{y=0} = \left.\frac{du(x, y)}{dy}\right|_{y=b} = 0\right)$, are versions 3, 4, and 6 in Fig. 1. Here,

$$\varphi_n^{(3)}(y) = \sin\left(\mu_n^{(3)} y\right), \quad \mu_n^{(3)} = \frac{(2n-1)\pi}{2b}, \quad n = 1, 2, ..., \infty; \tag{7}$$

$$u(x, y) = \frac{2}{b} \sum_{n=1}^{\infty} \overline{u_n}(x) \sin\left(\mu_n^{(3)} y\right). \tag{8}$$

Substituting into Eq. (2) the integral Fourier transform with finite limits 0 and b with respect to the variable y for each of the considered three types of boundary conditions yields an ordinary inhomogeneous differential equation

$$\frac{d^2 \overline{u_n}(x)}{dx^2} - \mu_n^2 \overline{u_n}(x) = -\overline{\rho_n}(x), \tag{9}$$

where $\overline{\rho_n}(x)$ is the nth transform of the function $\rho(x, y)$ with respect to the variable y for a specific form of the boundary conditions

$$\overline{\rho_n}(x) = \int_0^b \rho(x, y)\, \varphi_n(y)\, dy .$$

The form of the function $\overline{\rho_n}(x)$ depends on the boundary conditions on the boundaries $y = 0$ and $y = b$ and on the form of the function $\rho(x, y)$. With a rectangular shape of the heat source, for the considered three types of combinations of the boundary conditions on the boundaries $y = 0$ and $y = b$ the following expressions are the case for $\overline{\rho_n}(x)$:

for type 1 boundary conditions,

$$\overline{\rho_n^{(1)}}(x) = \frac{2P_i}{\mu_n^{(1)} l_i h_i Z\lambda} \sin\left(\mu_n^{(1)}\left(y_i + \frac{h_i}{2}\right)\right) \sin\left(\frac{\mu_n^{(1)} h_i}{2}\right) = \frac{2P_i}{\mu_n^{(1)} l_i h_i Z\lambda} \tilde{\rho}_n^{(1)} ; \tag{10}$$

for type 2 boundary conditions,

$$\overline{\rho_0^{(2)}}(x) = \frac{P_i}{l_i Z\lambda}, \quad \overline{\rho_n^{(2)}}(x) = \frac{2P_i}{\mu_n^{(2)} l_i h_i Z\lambda} \cos\left(\mu_n^{(2)}\left(y_i + \frac{h_i}{2}\right)\right) \sin\left(\frac{\mu_n^{(2)} h_i}{2}\right) = \frac{2P_i}{\mu_n^{(2)} l_i h_i Z\lambda} \tilde{\rho}_n^{(2)} \quad \text{for} \quad n > 0 ; \tag{11}$$

and for type 3 boundary conditions,

$$\overline{\rho_n^{(3)}}(x) = \frac{2P_i}{\mu_n^{(3)} l_i h_i Z\lambda} \sin\left(\mu_n^{(3)}\left(y_i + \frac{h_i}{2}\right)\right) \sin\left(\frac{\mu_n^{(3)} h_i}{2}\right) = \frac{2P_i}{\mu_n^{(3)} l_i h_i Z\lambda} \tilde{\rho}_n^{(3)} . \tag{12}$$

For all remaining values of x ($x < x_i$ or $x > x_i + l_i$), $\overline{\rho_n^{(1)}}(x) = \overline{\rho_n^{(2)}}(x) = \overline{\rho_n^{(3)}}(x) = 0$. In relations (10)–(12) $\tilde{\rho}_n^{(m)}$ ($m = 1, 2, 3$) denotes products of the pairs of trigonometric functions for perfect types of the boundary conditions. Equation (9) is reasonably solved using various approaches to mutually exclusive cases $\mu_n > 0$ and $\mu_n = 0$. For the case $\mu_n > 0$ ($\overline{u_n}(x)$, $n = 1, 2, \infty$) the solution of the inhomogeneous equation (9) is the sum of the general solution of the homogeneous equation

$$\frac{d^2 \overline{u_n}(x)}{dx^2} - \mu^2 \overline{u_n}(x) = 0 \tag{13}$$

and of the particular solution of the inhomogeneous equation (9). The general solution of the homogeneous equation is of the form

$$\overline{u_n}(x) = N_n \sinh(\mu_n x) + M_n \cosh(\mu_n x) . \tag{14}$$

The particular solution of the inhomogeneous equation (9) can be represented using the method of variation of constants [8] as

$$\overline{u_n}(x) = \frac{1}{\mu_n}\int_0^x \overline{\rho_n}(t)\sinh(\mu_n(t-x))\,dt\,. \tag{15}$$

Hence, the general solution of the inhomogeneous equation (9) represented by the sum of the general solution (14) of the homogeneous equation (13) and the particular solution (15) of the inhomogeneous equation (9) is of the form

$$\overline{u_n}(x) = N_n\sinh(\mu_n x) + M_n\cosh(\mu_n x) + \frac{1}{\mu_n}\int_0^x \overline{\rho_n}(t)\sinh(\mu_n(t-x))\,dt\,. \tag{16}$$

Since for versions 1–5 and 8 in Fig. 1 we have $u(x, y)\big|_{x=0} = 0$, which is possible only with $M_n = 0$, it should be assumed for them that $M_n = 0$. Then, the constant N_n can be determined from the boundary condition at $x = a$. Two variants of such conditions are possible.

1. Conductive heat removal occurs from the boundary $x = a$, and the temperature difference is $u(x, y)\big|_{x=a} = \overline{u_n}(x)\big|_{x=a} = 0$ (versions 2, 4, and 5 in Fig. 1). In this case from Eq. (16) we obtain

$$\overline{u_n}(x) = \frac{1}{\mu_n}\int_0^x \overline{\rho_n}(t)\sinh(\mu_n(t-x))\,dt - \frac{\sinh(\mu_n x)}{\mu_n\sinh(\mu_n a)}\int_0^a \overline{\rho_n}(t)\sinh(\mu_n(t-a))\,dt\,.$$

Integration of the last expression with account for the function $\overline{\rho_n}(t)$ for the considered boundary conditions gives

$$\overline{u_n^{(1)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\sinh(\mu_n x)\sinh(\mu_n(a-x_i-l_i/2))\sinh(\mu_n l_i/2)}{l_i h_i Z\lambda\mu_n^3\sinh(\mu_n a)} \quad \text{for} \quad x \le x_i\,, \tag{17}$$

$$\overline{u_n^{(1)}}(x) = \frac{2P_i\tilde{\rho}_n^{(m)}}{l_i h_i Z\lambda\mu_n^3}\left[\frac{2\sinh(\mu_n x)\sinh\left(\mu_n\left(a-x_i-\dfrac{l_i}{2}\right)\right)\sinh\left(\dfrac{\mu_n l_i}{2}\right)}{\sinh(\mu_n a)} - \cosh\Big(\mu_n(x_i-x)\Big)+1\right] \quad \text{for} \quad x_i \le x \le x_i + l_i\,, \tag{18}$$

$$\overline{u_n^{(1)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\sinh(\mu_n(a-x))\sinh(\mu_n(x_i+l_i/2))\sinh(\mu_n l_i/2)}{l_i h_i Z\lambda\mu_n^3\sinh(\mu_n a)} \quad \text{for} \quad x_i + l_i \le x\,. \tag{19}$$

2. There is no heat removal from the boundary $x = a$, i.e., the heat flux through the boundary $x = a$ is zero $\left(\left.\dfrac{du(x,y)}{dx}\right|_{x=a} = \left.\dfrac{d\overline{u_n}(x)}{dx}\right|_{x=a} = 0\right)$ (versions 1, 3, and 8 in Fig. 1). In this case, from Eq. (16) we find

$$\overline{u_n}(x) = \frac{1}{\mu_n}\int_0^x \overline{\rho_n}(t)\sinh(\mu_n(t-x))\,dt + \frac{\sinh(\mu_n x)}{\mu_n\cosh(\mu_n a)}\int_0^a \overline{\rho_n}(t)\cosh(\mu_n(t-a))\,dt\,.$$

Integration in this expression with account for the form of the function $\overline{\rho_n}(t)$ for these boundary conditions gives

$$\overline{u_n^{(2)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\sinh(\mu_n x)\cosh(\mu_n(a-x_i-l_i/2))\sinh(\mu_n l_i/2)}{l_i h_i Z\lambda\mu_n^3\cosh(\mu_n a)} \quad \text{for} \quad x \le x_i\,, \tag{20}$$

$$\overline{u_n^{(2)}}(x) = \frac{2P_i\tilde{\rho}_n^{(m)}}{l_ih_iZ\lambda\mu_n^3}\left[\frac{2\sinh(\mu_n x)\cosh\left(\mu_n\left(a - x_i - \frac{l_i}{2}\right)\right)\sinh\left(\frac{\mu_n l_i}{2}\right)}{\cosh(\mu_n a)} - \cosh\left(\mu_n(x_i - x)\right) + 1\right] \quad \text{for} \quad x_i \le x \le x_i + l_i\,, \tag{21}$$

$$\overline{u_n^{(2)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\cosh(\mu_n(a - x))\sinh(\mu_n(x_i + l_i/2))\sinh(\mu_n l_i/2)}{l_ih_iZ\lambda\mu_n^3\cosh(\mu_n a)} \quad \text{for} \quad x_i + l_i \le x\,. \tag{22}$$

For versions 6 and 7 in Fig. 1 where $\left.\frac{du(x, y)}{dx}\right|_{x=0} = \left.\frac{d\overline{u_n}(x)}{dx}\right|_{x=0} = 0$, which is possible at $N_n = 0$, the following variant of the expression $\overline{u_n}(x)$ is the case.

3. There is no heat removal from the boundary $x = a$ $\left(\left.\frac{du(x, y)}{dx}\right|_{x=a} = \left.\frac{d\overline{u_n}(x)}{dx}\right|_{x=a} = 0\right)$. Here, from Eq. (16) we obtain

$$\overline{u_n}(x) = \frac{1}{\mu_n}\int_0^x \overline{\rho_n}(t)\sinh(\mu_n(t - x))\,dt + \frac{\cosh(\mu_n x)}{\mu_n\sinh(\mu_n a)}\int_0^a \overline{\rho_n}(t)\cosh(\mu_n(t - a))\,dt\,.$$

For a subsequent discussion, of interest is integration only for the cases $x \le x_i$ and $x_i + l_i \le x$, since versions 6–8 in Fig. 1 are regarded as auxiliary ones, which provide the calculation of ΔT_{ij} with the intersection of projections of the ith and jth elements on the X axis for versions 1, 2, and 4 in Fig. 1. Allowance for the form of the function $\overline{\rho_n}(t)$ in Eqs. (10) and (11) gives

$$\overline{u_n^{(3)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\cosh(\mu_n x)\cosh(\mu_n(a - x_i - l_i/2))\sinh(\mu_n l_i/2)}{l_ih_iZ\lambda\mu_n^3\sinh(\mu_n a)} \quad \text{for} \quad x \le x_i\,, \tag{23}$$

$$\overline{u_n^{(3)}}(x) = \frac{4P_i\tilde{\rho}_n^{(m)}\cosh(\mu_n(a - x))\cosh(\mu_n(x_i + l_i/2))\sinh(\mu_n l_i/2)}{l_ih_iZ\lambda\mu_n^3\sinh(\mu_n a)} \quad \text{for} \quad x_i + l_i \le x\,. \tag{24}$$

The case of $\mu_n = 0$ ($\overline{u_0}(x)$) takes place only with the second type of boundary conditions on the boundaries $y = 0$ and $y = b$ and corresponds to $n = 0$ (see versions 1 and 2 in Fig. 1). Substituting Eqs. (5) and (11) into Eq. (9) yields

$$\frac{d^2\overline{u_{01}}(x)}{dx^2} = 0 \quad \text{for} \quad x \le x_i\,, \quad \frac{d^2\overline{u_{02}}(x)}{dx^2} = -\frac{P_i}{l_iZ\lambda} \quad \text{for} \quad x_i \le x \le x_i + l_i \quad \text{and} \quad \frac{d^2\overline{u_{03}}(x)}{dx^2} = 0 \quad \text{for} \quad x \ge x_i + l_i\,,$$

where $\overline{u_{01}}(x)$, $\overline{u_{02}}(x)$, and $\overline{u_{03}}(x)$ are expressions of $\overline{u_0}(x)$ for appropriate regions of the plate along the X axis.

Solutions of these equations are obtained using the boundary conditions at $x = 0$ and $x = a$ and also the conditions of constancy of temperature differences and continuity of heat fluxes at all points of vertical sections of the plate at $x = x_i$ and $x = x_i + l_i$, which are of the form

$$\left.\overline{u_{01}}(x)\right|_{x=x_i} = \left.\overline{u_{02}}(x)\right|_{x=x_i}\,; \quad \left.\overline{u_{02}}(x)\right|_{x=x_i+l_i} = \left.\overline{u_{03}}(x)\right|_{x=x_i+l_i}\,;$$

$$\left.\frac{d\overline{u_{01}}(x)}{dx}\right|_{x=x_i}=\left.\frac{d\overline{u_{02}}(x)}{dx}\right|_{x=x_i};\quad \left.\frac{d\overline{u_{02}}(x)}{dx}\right|_{x=x_i+l_i}=\left.\frac{d\overline{u_{03}}(x)}{dx}\right|_{x=x_i+l_i}.$$

For $x = 0$ in versions 1 and 2 in Fig. 1, $\overline{u_{01}}(x)\big|_{x=0} = 0$, and for $x = a$ the following two types of the boundary conditions are possible.

1. $\overline{u_{03}}(x)\big|_{x=a} = 0$ corresponds to version 2 in Fig. 1 for the second type of the boundary conditions at $y = 0$ and $y = b$. Here we obtain

$$\overline{u_0^{(1)}}=\frac{P_i}{Z\lambda}\left(1-\frac{2x_i+l_i}{2a}\right)x \quad \text{for} \quad x\le x_i\,, \tag{25}$$

$$\overline{u_0^{(1)}}=\frac{P_i}{Z\lambda}\left[\left(1-\frac{2x_i+l_i}{2a}\right)x-\frac{(x-x_i)^2}{2l_i}\right] \quad \text{for} \quad x_i\le x\le x_i+l_i\,, \tag{26}$$

$$\overline{u_0^{(1)}}=\frac{P_i}{Z\lambda}\left(x_i+\frac{l_i}{2}\right)\left(1-\frac{x}{2a}\right) \quad \text{for} \quad x_i+l_i\le x\,. \tag{27}$$

2. $\left.\frac{d\overline{u_{03}}(x)}{dx}\right|_{x=a} = 0$ corresponds to version 1 in Fig. 1 for the second type of boundary conditions at $y = 0$ and $y = b$. Here we obtain

$$\overline{u_0^{(2)}}=\frac{P_i}{Z\lambda}x \quad \text{for} \quad x\le x_i\,; \tag{28}$$

$$\overline{u_0^{(2)}}=\frac{P_i}{Z\lambda}\left[x-\frac{(x-x_i)^2}{2l_i}\right] \quad \text{for} \quad x_i\le x\le x_i+l_i\,; \tag{29}$$

$$\overline{u_0^{(2)}}=\frac{P_i}{Z\lambda}\left(x_i+\frac{l_i}{2}\right) \quad \text{for} \quad x_i+l_i\le x\,. \tag{30}$$

Thus, expressions (4), (6), and (8) with substitutions of the values of $\overline{u_n}(x)$ from expressions (17)–(22) and (25)–(30) are solutions of Eq. (2) for versions 1–5 and 8 in Fig. 1 for $0 \le x \le a$, and with substitutions of the values of $\overline{u_n}(x)$ from Eqs. (23) and (24) for $x \le x_i$ and $x_i + l_i \le x$, they are solutions of the above equation for versions 6 and 7 in Fig. 1.

Determination of the Average Temperature Difference for Rectangular Elements. The calculation of the average temperature difference ΔT_i for the ith element incorporates two problems, namely, internal and external. The internal problem is taken to mean the calculation of the average temperature difference caused by the heating element per second (the calculation of ΔT_{ii} in Eq. (1)). By the external problem is meant the calculation of the average temperature difference caused by other heating elements (the calculation of ΔT_{ij} at $i \ne j$ in Eq. (1)). If $U_i(x, y)$ is used to denote the temperature difference $u(x, y)$ from the ith element at the point (x, y), where $i = 1, 2, 3, ..., N$, then the expression for the internal problem is

$$\Delta T_{ii}=\frac{1}{l_i h_i}\int\limits_{x_i}^{x_i+l_i}\int\limits_{y_i}^{y_i+h_i}U_i(x,y)\,dxdy\,.$$

Correspondingly, for the external problem

$$\Delta T_{ij} = \frac{1}{l_i h_i} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{x_i}^{x_i+l_i} \int_{y_i}^{y_i+h_i} U_j\,(x, y)\, dxdy\,.$$

Using Eqs. (4), (6), and (8), as well as the expressions obtained previously for $\overline{u_n^{(p)}}(x)$ and $\varphi_n^{(m)}(y)$, and denoting them respectively by $u_{in}^{(p)}(x)$ and $\varphi_{in}^{(m)}(y)$ for an element with the subscript i, it is possible to determine the average temperature difference for the internal and external problems.

It is not difficult to see that for the internal problem

$$\Delta T_{ii} = \frac{1}{l_i h_i}\left(\frac{1}{b} \int_{x_i}^{x_i+l_i} \overline{u_{i0}^{(p)}}\,(x)\, dx \int_{y_i}^{y_i+h_i} dy + \frac{2}{b} \sum_{n=1}^{\infty} \int_{x_i}^{x_i+l_i} \overline{u_{in}^{(p)}}\,(x)\, dx \int_{y_i}^{y_i+h_i} \varphi_{in}^{(m)}\,(y)\, dy \right). \tag{31}$$

Here, integration is carried out over the area of the ith element, and the functions $\overline{u_{i0}^{(p)}}(x)$, $\overline{u_{in}^{(p)}}(x)$, and $\varphi_{in}^{(m)}(y)$ correspond to the internal problem in the intervals $x_i \leq x \leq x_i + l_i$ and $y_i \leq y \leq y_i + h_i$.

For the external problem we have

$$\Delta T_{ij} = \frac{1}{l_i h_i} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{1}{b} \int_{x_i}^{x_i+l_i} \overline{u_{j0}^{(p)}}\,(x)\, dx \int_{y_i}^{y_i+h_i} dy + \frac{2}{b} \sum_{n=1}^{\infty} \int_{x_i}^{x_i+l_i} \overline{u_{jn}^{(p)}}\,(x)\, dx \int_{y_i}^{y_i+h_i} \varphi_{jn}^{(m)}\,(y)\, dy \right). \tag{32}$$

Here, integration is performed over the region of the ith element $x_i \leq x \leq x_i + l_i$, $y_i \leq y \leq y_i + h_i$ and as the functions $u_{j0}^{(p)}(x)$, $u_{jn}^{(p)}(x)$, and $\varphi_{jn}^{(m)}(y)$ use is made of $u_0^{(p)}(x)$, $u_n^{(p)}(x)$, and $\varphi_n^{(m)}(y)$ corresponding to the jth elements whose left lower angles are located at the points (x_j, y_j) and whose dimensions are $l_j \times h_j$. From Eqs. (4), (6), and (8) it is seen that, in expressions (31) and (32), $u_{i0}^{(p)}(x) = u_{j0}^{(p)}(x) = 0$ for all heat removal versions in Fig. 1, except versions 1 and 2.

If in expression (32) use is made of $\overline{u_{jn}^{(p)}}(x)$ from Eqs. (17), (19), (20), (22), (25), (27), (28), and (30), then projections of the ith and jth elements on the X axis should not intersect. In the case of their intersection for versions 3 and 5 in Fig. 1 it is necessary to change coordinates ($x := y^{(\mathrm{in})}$, $y := x^{(\mathrm{in})}$, $x_i := y_i^{(\mathrm{in})}$, $y_i := x_i^{(\mathrm{in})}$, $l_i := h_i^{(\mathrm{in})}$, $h_i := l_i^{(\mathrm{in})}$, $a := b^{(\mathrm{in})}$, and $b := a^{(\mathrm{in})}$. Here, the superscript (in) denotes initial values of pertinent quantities before the change of coordinates. In the case of intersection of such projections for versions 1, 2, and 4 in Fig. 1, through the indicated transformations of coordinates they should be brought correspondingly to auxiliary versions 6, 7, and 8 in Fig. 1, and $u_{jn}^{(p)}(x)$ from Eqs. (23) or (24) should be used.

Determination of the Average Temperature Differences on Rectangular Elements for Various Forms of Boundary Conditions. As stated above, the "thermal construction" of printed circuit boards with predominant conductive heat removal through metal plates of the substrates or cores allows a different combination of the edges of such plates, from which heat is removed. With one-, two-, three-, and four-way heat removal the total number of such versions is 15. It is not difficult to see that without loss of generality the number of these versions can be reduced to the basic first five ones presented in Fig. 1, and the other 10 versions are readily obtained from the basic five ones through a simple transformation of coordinates.

Figure 1 shows heating and passive elements arranged on the plate. The average temperature difference for each element can be determined from expression (1). Here, if the jth element is a heating one, $\Delta T_{ij} > 0$, and if it is passive, $\Delta T_{ij} = 0$. If the ith element is a heating one, $\Delta T_{ii} > 0$, and if it is passive, $\Delta T_{ii} = 0$.

Each version of heat removal from the plate edges is characterized by the combination of the values of m (1, 2, or 3) and p (1, 2, or 3), which specify the combination of boundary conditions on these edges and determine the

TABLE 1. Parameters of Expressions for ΔT_{ii} and ΔT_{ij}

Version of heat removal from the plate (see Fig. 1)	m	p	k	f_1	f_2	f_{ii}	Position of the ith and jth elements					
							$x_i + l_i \le x_j$, $A_i = x_i + l_i/2$, $A_j = a - x_j - l_j/2$			$x_j + l_j \le x_i$, $A_i = a - x_i - l_i/2$, $A_j = x_j + l_j/2$		
							f_{ij}	f_i	f_j	f_{ij}	f_i	f_j
1	2	2	n	cos	cosh	F_1	F_3	sinh	cosh	F_5	cosh	sinh
2	2	1	n	cos	sinh	F_2	F_4	sinh	sinh	F_6	sinh	sinh
3	3	2	$n-1/2$	sin	cosh	0	0	sinh	cosh	0	cosh	sinh
4	3	1	$n-1/2$	sin	sinh	0	0	sinh	sinh	0	sinh	sinh
5	1	1	n	sin	sinh	0	0	sinh	sinh	0	sinh	sinh
6	3	3	$n-1/2$	sin	sinh	—	0	cosh	cosh	0	cosh	cosh
7	1	3	n	sin	sinh	—	0	cosh	cosh	0	cosh	cosh
8	1	2	n	sin	cosh	—	0	sinh	cosh	0	cosh	cosh

$$F_1 = \frac{P_i}{Z\lambda b}\left(x_i + \frac{l_i}{3}\right) \qquad F_2 = \frac{P_i}{Z\lambda b}\left(x_i + \frac{l_i}{3} - \frac{(2x_i + l_i)^2}{4a}\right)$$

$$F_3 = \frac{P_j}{Z\lambda b}\left(x_i + \frac{l_i}{2}\right) \qquad F_4 = \frac{P_j}{Z\lambda b}\left(1 - \frac{2x_j + l_j}{2a}\right)\left(x_i + \frac{l_i}{2}\right)$$

$$F_5 = \frac{P_j}{Z\lambda b}\left(x_j + \frac{l_j}{2}\right) \qquad F_6 = \frac{P_j}{Z\lambda b}\left(x_j + \frac{l_j}{2}\right)\left(1 - \frac{2x_j + l_i}{2a}\right)$$

form of the expressions for $\varphi_n^{(m)}(y)$ and μ_n from Eqs. (3), (5), and (7), $\tilde{\rho}_n^{(m)}$ from Eqs. (10–(12), and $\overline{u_n^{(p)}}(x)$ from Eqs. (17)–(30). For the basic first five versions of heat removal from the plate edges (see Fig. 1), the expression of ΔT_{ii} obtained from Eq. (31) using Eqs. (4), (6), and (8) can be written in general form as

$$\Delta T_{ii} = f_{ii} + \frac{8b^4 P_i}{\pi^5 l_i^2 h_i^2 Z\lambda} \sum_{n=1}^{\infty} \left\{ \frac{f_1^2\left(\frac{k\pi\,(y_i + h_i/2)}{b}\right)\sin^2\left(\frac{k\pi h_i}{2b}\right)}{k^5} \times \right.$$

$$\left. \times \left[\frac{4f_2\left(\frac{k\pi\,(a - x_i - l_i/2)}{b}\right)\sinh\left(\frac{k\pi\,(x_i + l_i/2)}{b}\right)\sinh^2\left(\frac{k\pi l_i}{2b}\right)}{f_2\left(\frac{k\pi a}{b}\right)} - \sinh\left(\frac{k\pi l_i}{b}\right) + \frac{k\pi l_i}{b} \right] \right\},$$

and the general expression of ΔT_{ij} for all eight versions of heat removal can be represented as

$$\Delta T_{ij} = f_{ij} + \frac{32 b^4 P_j}{\pi^5 l_i h_i l_j h_j Z\lambda} \sum_{n=1}^{\infty} \left\{ \frac{f_1\left(\frac{k\pi\,(y_i + h_i/2)}{b}\right) f_1\left(\frac{k\pi\,(y_j + h_j/2)}{b}\right)\sin\left(\frac{k\pi h_i}{2b}\right)\sin\left(\frac{k\pi h_j}{2b}\right)}{k^5} \times \right.$$

$$\times \frac{f_i(k\pi A_i)\, f_j(k\pi A_j)\, \sinh\left(\frac{k\pi l_i}{2b}\right)\sinh\left(\frac{k\pi l_j}{2b}\right)}{f_2\left(\frac{k\pi a}{b}\right)}\Bigg\} \quad \text{for} \quad x_i + l_i \le x_j \quad \text{or} \quad x_j + l_j \le x_i \,.$$

Here, the expressions for k, A_i, and A_j and the form of functions f_{ii}, f_{ij}, f_1, f_2, f_i, and f_j are determined by the version of heat removal from the plate edges. Table 1 presents the values of m and p, the expressions for k, A_i, and A_j, and also the form of functions f_{ii}, f_{ij}, f_1, f_2, f_i, and f_j in general expressions of ΔT_{ii} and ΔT_{ij} for various versions in Fig. 1.

When general expressions of ΔT_{ii} and ΔT_{ij} are used for versions of heat removal from the plate other than the first five versions in Fig. 1, the transformation of coordinates is required because of the "turning" of a corresponding version on Fig. 1. In the case of intersection of projections of the ith and jth elements on the axis X for ΔT_{ij}, the second transformation of coordinates (for versions 1, 2, and 4 in Fig. 1) is needed, which results in versions 6–8.

With a specific version of heat removal and use of Eq. (1), general expressions for ΔT_{ii} and ΔT_{ij} allow determination of the average temperature difference ΔT_i for each ith passive and heating element. The main difficulty in calculating ΔT_{ii} and ΔT_{ij} thus presented lies in the summation of series. Their convergence may be improved using the results of [5, 9].

CONCLUSIONS

1. Expressions have been obtained for distributions of temperature differences for any points on a thin heat-conducting rectangular plate from a single rectangular heating element in the body of the plate under various boundary conditions.

2. An approach has been formulated, and expressions presented, for determining the temperature difference average over the element area in the presence of more than one rectangular heating element in the body of a thin heat-conducting plate.

3. The general form of calculational expressions for the average temperature differences on rectangular elements has been obtained for the basic five versions of heat removal from the plate edges, which can be extended to all 15 possible such versions corresponding to actual structures of printed circuit boards with conductive heat removal.

4. Under the adopted assumptions, the examined problem has a relatively simple analytical solution for various combinations of the 1st- and 2nd-kind boundary conditions.

NOTATION

A_i and A_j, form of the factor of arguments of the functions f_i and f_j with a specific relative position of the ith and jth elements; a, plate dimension along the X axis; b, plate dimension along the Y axis; F_1–F_6, expressions of the zero term of the series for solutions of the internal and external problems; f_{ii}, form of the zero term of the series of ΔT_{ii} for a specific form of boundary conditions; f_{ij}, form of the zero term of the series of ΔT_{ij} for a specific form of boundary conditions; f_1, f_2, f_i, and f_j, form of the trigonometric and hyperbolic functions for a specific form of boundary conditions; h_i, dimension of the ith rectangular element along the Y axis; k, value of μ_n in the general expression of solutions for a specific form of boundary conditions; l_i, dimension of the ith rectangular element along the X axis; M_n, constant at the function sinh in the general form of the function $\overline{u_n}(x)$; m, number of the version of boundary conditions on horizontal edges of the plate; N, number of EREs on the printed circuit board; N_n, constant at the function cosh in the general form of the function $\overline{u_n}(x)$; p, number of the version of boundary conditions on vertical edges of the plate; $\overline{u_{01}}(x)$, $\overline{u_{02}}(x)$, and $\overline{u_{03}}(x)$, form of the function $\overline{u_0}(x)$ for regions $x \le x_i$, $x_i \le x \le x_i + l_i$, and $x \ge x_i$ respectively; $\overline{u_n}(x)$, nth transform of integral transformation of the function $u(x, y)$ with respect to the variable y; $\overline{u_n^{(p)}}(x)$, form of the function $\overline{u_n}(x)$ for the pth version of boundary conditions on vertical edges of the plate; $\overline{u_{in}^{(p)}}(x)$, value of $\overline{u_n^{(p)}}(x)$ for the ith element; $U_i(x, y)$ and $u(x, y)$, temperature difference from the ith element at the point (x, y); x and y, coordinates of the point (x, y) in a rectangular coordinate system with origin at the left lower angle of the plate; x_i and y_i, coordinates of the left lower angle of the ith rectangular element; Z, plate thickness; ΔT_i, temperature difference for the ith element with all N elements placed on the printed circuit board; ΔT_{ij}, temperature difference from the jth element in the region of the ith element only jth element on the printed circuit board; $\varphi_n(y)$, kernel of the integral transform; $\varphi_n^{(m)}(y)$, form of the function $\varphi_n(y)$, for the m type of boundary conditions on horizontal edges of the plate; $\varphi_{in}^{(m)}(y)$, value of $\varphi_n^{(m)}(y)$ of the ith element; λ, specific thermal conductivity of the plate; μ_n, nth plate value of the coefficient at y in the function $\varphi_n(y)$; $\mu_n^{(m)}(y)$, nth value of the coefficient at y in the function $\varphi_n^{(m)}(y)$ for the mth type of boundary conditions on horizontal edges of the plate; $\overline{\rho_n}(x)$, Fourier transform of the function $\rho(x, y)$; $\tilde{\rho}_n^{(m)}$, products of pairs of the trigonometric functions for expressions of $\rho_n^{(m)}(x)$; $\rho_n^{(m)}(x)$, form of the function $\overline{\rho_n}(x)$ for the mth type of boundary conditions on horizontal edges of the plate; $\rho(x, y)$, surface density of the heat power of the ith heat source at any point (x, y) divided into the specific thermal conductivity of the plate material.

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